$\mathcal{HEAVEN}$ & $\mathcal{HELL}$

by

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Abstract

This paper describes how Escher's **Heaven and Hell**, in a precise sense, lies over every possible 3-dimensional world: *Encoded in its symmetries lies the data to construct all 3-dimensional manifolds*. This universality is akin to, but different from, that of Hilden–Lozano–Montesinos–Whitten.

'Representations are nothing but images

*of spiritual * things in natural ones,*

and when the former are rightly represented

in the latter, then the two correspond.'

– Emanuel Swedenborg,

Arcana Cœlestia, 4044.

1. Heaven & Hell

We describe some hitherto unknown remarkable properties of the famous *Circle Limit IV* by the Dutch graphic artist M. C. Escher. Much of Escher's work has been interactive with mathematical research, but in some ways *Circle Limit IV*, alternatively named '**Heaven and Hell**', has particular significance, having become one of the few full-size posters available, but more importantly, being the choice symbol for the International Congress on M.C. Escher in Rome, 1985 [Es2]. In Escher's own words:

"*The disc is divided into six sections in which, turn and turn about, the angels on a black background and then the devils on a white one, gain the upper hand. In this way, heaven and hell change place six times. In the intermediate, "earthly" stages, they are equivalent.*"

— Circle Limit IV [Es1]

There is 2-fold symmetry of the standard Good/Evil archetypal dichotomy manifest in the interlocking portrayal of Angels and Devils, and also 3-fold symmetry, the centre of the picture being the common point of three Devils and three Angels.

* *i.e.'intellectual' – see A.C., 3886*

2. Symmetries, subgroups and surfaces

We begin with an elementary description of the (minimal) hyperbolic geometry required, using the Poincaré disc model $\mathcal{D}^2$.

The hyperbolic plane $\mathcal{D}^2$ is the *interior* of the unit disc, and *geodesics* are arcs of circles meeting the boundary circle orthogonally.

Inversions of the Euclidean plane in the circles defining geodesics preserve $\mathcal{D}^2$, and are hyperbolic isometries known as *reflections*.

A *regular (hyperbolic) polygon* has all angles equal, and edges made up of geodesic arcs of equal length.

In order to describe Heaven and Hell as a tesselation $\mathcal{H}$ of the hyperbolic plane, we summarize properties playing a role in its construction:

1. Circular arcs may be drawn through the points where wingtips meet, passing across the outspread wings of the angels. These divide the picture into hyperbolic 'hexagons', four of which meet at each vertex.
2. The feet of three angels and three devils meet at the centre of each hexagon. Hence the intrinsic 6-fold symmetry of the abstract mathematical tesselation has been broken to a 3-fold symmetry, whereby we preclude rotations interchanging angels and devils. (For a slightly more courageous viewpoint, see [Le].)
3. The Heaven/Hell dichotomy of the central hexagon can be abstractly embodied by *two-colouring* the geodesic arcs of the hexagon, thereby creating two intersecting networks of disjoint geodesic arcs on the disc.
4. If we 'invert' the picture through any of these circular arcs, we interchange angels and devils, and simultaneously reverse the orientation of the disc. The 2-colouring

of geodesics is preserved by inversion.

The geometry of the hyperbolic plane is infinitesimally Euclidean. If we uniformly expand a regular hexagon at the centre of $\mathcal{D}^2$, its interior vertex angles decrease continuously from $2\pi/3$ to 0, as vertices approach the circle at infinity. There is exactly one regular hexagon R_6 in $\mathcal{D}^2$ with interior vertex angle $\frac{\pi}{2}$.

This has a particularly simple direct construction: Take an arrangement of seven circles of equal radius 1, arranged so that six circles surround and tangentially touch the seventh. Keeping the centres of these seven circles fixed, simultaneously and uniformly enlarge them until each has radius $\sqrt{2}$. A simple calculation shows they are mutually perpendicular at intersections. Now shrink the configuration concentrically about the centre point, until the central circle has radius 1. The hexagon in the centre is the desired R_6.

Equivalently, inscribe a regular hexagon in a circle of radius $\sqrt{2}$, and then draw seven circles of radius one about the vertices of the hexagon and the centre of the circle. Analogous descriptions work in the construction of other regular polygons.

Reflections σ_i in 6 cyclically labelled edges e_i of R_6 generate a *Coxeter group* Γ^3, with fundamental domain R_6 ([C-M]).

$$\Gamma^3 =< \sigma_i,\ i = 1, \dots,\ 6 \ \mid\ \sigma_i^2 = 1,\ (\sigma_i\sigma_{i+1})^2 = 1,\ (\text{mod } 6) > .$$

The images of R_6, under the elements of the group, tesselate $\mathcal{D}^2$ with isometric hexagons, four meeting at each vertex.

Colour the edges of R_6 alternatively red and black, with convention, edge e_{2s} is red. Acting by Γ^3 tesselates $\mathcal{D}^2$, all hexagons having alternating red and black edges. The plane is divided up by two families of disjoint geodesics, one red, the other black, setwise invariant under Γ^3.

Denote by $\mathcal{H}$ the hyperbolic plane, together with this pattern. $\mathcal{H}$ is the mathematical abstraction underlying Escher's Heaven & Hell.

Since the reflections in edges of hexagons reverse orientation, and simultaneously interchange angels and devils, we have

Lemma 2.1. *An element $g \in \Gamma^3$ preserves orientation iff it preserves angels and devils.*

Symmetries of $\mathcal{H}$ with fixed points are either

(a) reflections in a geodesic,

(b) rotations about the centre of a hexagon,

(c) rotations about a vertex of a hexagon, or

(d) rotations about the mid-point of an edge of a hexagon.

Definition 2.2. A torsion-free, orientation-preserving subgroup H of Γ^3 will be called *heavenly*.

Remark 2.3. This terminology seems more appropriate than the over-used 'good', which in its own right is better adapted to the situation here considered than is usually the case!

A heavenly subgroup H of *finite index* in Γ^3 acts freely and discretely on $\mathcal{H}$ with fundamental domain a finite union of hexagons. The quotient is a closed orientable surface F_H, with metric of constant curvature -1, and marked by two families, red and black, of disjoint closed geodesics, meeting at right-angles and decomposing F_H into regular hyperbolic hexagons.

Definition 2.4. A *heavenly surface* F_M is an orientable surface F, together with a *marking* M, consisting of two families, H_r (red) and H_b (black), each of disjoint non-trivial simple closed curves on F, decomposing F into a union of hexagons.

Thus F_H is canonically a heavenly surface. Conversely, a heavenly surface can be given a canonical metric of constant curvature -1 by declaring each hexagon to be a regular hyperbolic right-angled hexagon. This gives immediately:

Lemma 2.5. *There is a 1:1 correspondence between closed heavenly surfaces F_M and conjugacy classes of finite-index heavenly subgroups in Γ^3. If F_M has genus $k+1 \geq 2$, the corresponding subgroup index is $4k$. Hence heavenly subgroups have index divisible by 4.*

The genus can be calculated easily, since $4k$ hexagons glue together to give $12k$ edges and $6k$ vertices, and thus the Euler characteristic of the quotient is $2-2g=-2k$. Any orientable surface has even Euler characteristic.

3. Constructing 3-dimensional spaces from subgroups

We associate a 3-dimensional space to each heavenly subgroup:

Let H be heavenly, of finite index. On F_H, take disjoint red annular neighbourhoods $A_\gamma \cong S^1 \times I$, one for each red curve γ. Similarly, take a disjoint collection of black annuli. Construct $F_H \times [-1,1]$, and consider copies of all red annuli on $F_H \times \{-1\}$, and all black annuli on $F_H \times \{1\}$.

Attach 2-handles $D^2 \times I$ along these annuli on $F_H \times \{\pm 1\}$, obtaining an orientable 3-manifold with disconnected compact boundary.

Now abstractly cone boundary components, each to a distinct point. The resulting 3-dimensional space M^3_H is a 3-dimensional manifold, except possibly at finitely many points, where it has local structure the cone on some closed orientable surface of genus ≥ 1.

Definition 3.1. We call H *r-filling* if the family H_r of embedded red geodesic curves decomposes F_H into a (possibly disconnected) planar surface. Similarly, we call it *b-filling* if the black family H_b has this property. We denote the collection of all r-filling finite index subgroups by $\mathcal{F}^3_r$, and the b-filling ones by $\mathcal{F}^3_b$.

Observation 3.2. There is an obvious isomorphism $\mathcal{F}^3_r \cong \mathcal{F}^3_b$, by changing colours.

Denote by $\mathcal{G}^3$ the set of equivalence classes of heavenly subgroups $H < \Gamma^3$, two subgroups being equivalent if they are conjugate in Γ^3. Let $[H] \in \mathcal{G}^3$ denote the corresponding equivalence class. Let $\mathcal{G}^3_r$ and $\mathcal{G}^3_b$ denote the corresponding equivalence classes of filling subgroups. Consider heavenly subgroups $H,\ G$.

Proposition 3.3. *If* $[H] = [G]$, *then* $M^3_H \cong M^3_G$.

Proof. Conjugate subgroups have equivalent markings: there is an isometry $F_H \longrightarrow F_G$ carrying one marked structure to the other. This induces a homeomorphism of the product, which extends over the 2-handles, thence over the cones.

Theorem 3.4. M^3_H *is a closed orientable 3-manifold iff* $[H] \in \mathcal{G}^3 \equiv \mathcal{G}^3_r \cap \mathcal{G}^3_b$.

Proof. We obtain a closed orientable 3-manifold iff there are no singular points, iff all cones are on 2-spheres. After adding 2-handles to $F_H \times [-1, 1]$, all boundary components are 2-spheres iff $[H] \in \mathcal{G}^3$.

Proposition 3.5. *If* M^3_H *is a closed 3-manifold, the embedded surface* $F_H \equiv F_H \times \{0\}$ *separates* M^3_H *into two handlebodies. Thus* F_H *is a Heegard surface.*

Proof. The last statement is by definition of a Heegard surface.

4. Constructing subgroups from 3-dimensional spaces

Having associated spaces to heavenly subgroups, we show how to associate heavenly subgroups to certain spaces.

Consider a compact, convex body $\mathcal{B}^3$ in Euclidean space $\mathbf{R^3}$, defined as the intersection of finitely many half-spaces. Edges and vertices of the boundary of such a polyhedron derive from the intersection of those planes in $\mathbf{R^3}$ corresponding to the half-spaces. These edges and vertices define a connected graph $\mathcal{G}_\mathcal{B}$ on the 2-sphere boundary of the body.

Generically, a set of planes is in general position: at most 3 planes intersect at any given point, and so all vertices are of degree 3. Thus we expect that the graph $\mathcal{G}_\mathcal{B}$ of a randomly chosen convex body will have all vertices of degree 3. This motivates

Definition 4.1. A *generic (abstract) polyhedron* is a 3-ball $\mathcal{B}$ on whose boundary is embedded a connected, finite graph $\mathcal{G}_{\mathcal{B}}$, all of whose vertices are of degree 3. Regions of $\partial\mathcal{B} \cong S^2$ complementary to $\mathcal{G}_{\mathcal{B}}$ are called *faces*.

Remark 4.2. Faces are topologically discs, with a natural structure of an n-gon, for some $n \geq 1$. Moreover, these polygonal boundaries need not be embedded on S^2: we do not exclude an edge beginning and ending at the same vertex, nor edges both sides of which are the same face. Such edges are counted twice for the corresponding polygon.

In general these structures do not all arise as convex bodies in $\mathbf{R^3}$ as described above. However, by drawing any degree-3 graph on the boundary of the standard ball in $\mathbf{R^3}$, we see that every generic polyhedron admits two possible orientations. Each face inherits an orientation by taking the outward-pointing normal.

Let $\mathcal{P}_3$ denote the set of generic (abstract) oriented polyhedra.

Let $X_1,\ X_2 \in \mathcal{P}_3$, where possibly $X_1 = X_2$. Consider two distinct faces $F_{1i} \in X_1$, and $F_{2j} \in X_2$. If these are both n-gons, we can identify these in any of $2n$ canonical ways, up to isotopy, corresponding to the elements of the dihedral group $\mathcal{D}_{2n}$.

More generally, we may take a finite set of generic polyhedra, and pairwise identify faces to obtain a *generic complex*. The complement of the vertices of such a complex is a 3-dimensional manifold. This will be orientable iff we require all identifications to reverse the canonical orientation on faces.

Definition 4.3. We denote by $\mathcal{PC}_3$ the set of closed orientable complexes obtained by taking a disjoint union of finitely-many oriented elements (not necessarily distinct) of $\mathcal{P}_3$, such that all faces are paired and identified as above.

Remark 4.4. Not all elements of $\mathcal{PC}_3$ are closed 3-manifolds, since neighbourhoods of vertices are cones on closed surfaces which need not be S^2. In order for $Z \in \mathcal{PC}_3$ to be a 3-manifold, it is necessary and sufficient for its Euler characteristic to vanish. (See Thurston [Th1].)

Whenever we form an element of $\mathcal{PC}_3$, we obtain a description analogous to a *Heegard splitting*. This arises naturally via duality.

We define a natural map $\tau_* : \mathcal{P}_3 \longrightarrow \mathcal{P}_3$. This amounts to taking the dual and truncating all vertices:

Suppose $X \in \mathcal{P}_3$ is defined by the graph $\mathcal{G}_X$. Construct the *dual* graph $\mathcal{G}_X^*$. In general this will not have all vertices of degree three. We correct this by abstractly *truncating* each vertex, replacing the vertex v of degree k with a k-gon, with edges originally incident with v now incident at a vertex of the corresponding polygon. This results in a new graph $\mathcal{G}_{\tau_*(X)}$, which defines an element $\tau_*(X) \in \mathcal{P}_3$.

Example 4.5. Starting with X the cube, we first obtain an octahedron, and finally a truncated octahedron.

Claim 4.6. *The edges of $\mathcal{G}_{\tau_*(X)}$ admit a canonical 2-colouring.*

Proof. Colour all of the edges of $\mathcal{G}_X^*$ red, and all edges of polygons introduced in truncation, black.

Proposition 4.7. *The canonical 2-colouring of the graph decomposes S^2 into black polygons, and hexagons whose edges are alternatively coloured red and black. Hexagons abut hexagons along red edges.*

In terms of geometric realisations as convex sets in $\mathcal{R}^3$, if we inscribe the dual X^* of X *inside* X, expand it slightly so that its vertices protrude through the interiors of the faces of X to obtain X', the polyhedron $\tau_*(X)$ has geometric realisation as a convex polyhedron in $\mathcal{R}^3$ given by $X \cap X'$.

Remark 4.8. The map τ_* is clearly not onto. However,

Proposition 4.9. *The map τ_* is injective with respect to colourings of the image.*

Proof. The procedure is reversible: given a polyhedron $Y \in \mathcal{P}_3$, with faces marked as above, there is a unique $X \in \mathcal{P}_3$ such that $Y = \tau_*(X)$, obtained by reversing the truncation and taking the dual. The 2-colouring is necessary to identify those faces arising from truncation.

Consider $Z \in \mathcal{PC}_3$, obtained by identifying all faces of $\{X_1, \ldots, X_n\} \subset \mathcal{P}_3$. Inside Z we see $\tau_*(Z)$, obtained by identifying the respective $\tau_*(X_i)$. Observe that $\tau_*(Z)$ is a *handlebody*, with boundary some genus g surface F_Z.

Proposition 4.10. *The surface F_Z is heavenly.*

Proof. The boundary of each $\tau_*(X_i)$ consists of black polygons and 2-coloured hexagons. Identifications are carried out on pairs of black polygons. Each vertex has two black edges and one red incident. Two hexagons meet at each vertex of $\mathcal{G}_{\tau_*(X_i)}$, and so after identifications, we find four hexagons meeting at each vertex of F_Z. Through each vertex passes a red polygonal curve, and a black one, giving the desired marking by simple closed curves.

An immediate consequence of the construction is

Proposition 4.11. *Let $H < \Gamma^3$ be representative of the conjugacy class of subgroups corresponding to F_Z. Then $[H] \in \mathcal{G}_b^3$.*

5. Correspondences and the universality of Heaven and Hell

Definition 5.1. Consider complexes $Z_1,\ Z_2 \in \mathcal{PC}_3$ to be *equivalent* if there is a homeomorphism $Z_1 \longrightarrow Z_2$ preserving vertices, edges, faces and 3-balls. Denote the corresponding equivalence class by $[Z_1]$.

Theorem 5.2. *There is a 1:1 correspondence between equivalence classes $[Z]$ of generic closed complexes, and equivalence classes of b-filling heavenly subgroups of finite index.*

Proof. For one direction, equivalent complexes give rise to equivalent marked surfaces, which by Proposition 2.5 correspond to equivalence classes of heavenly subgroups of finite index. These are of course b-filling.

For the converse, verify that $H \in \mathcal{G}_b^3$ determines a canonical $Z \in \mathcal{PC}_3$. We show that the appropriate Z is homeomorphic to M_H^3.

First cut F_H open along the curves in H_b. Add 2-dimensional discs to the resulting planar surfaces along the boundary curves to obtain some number of 2-dimensional spheres. Each such disc becomes a polygon, since the circle to which it is attached has some finite number of intersections with red circles. Each resulting 2-sphere has the structure of $\tau_*(X)$, for a uniquely determined $X \in \mathcal{P}_3$. Now consider glueing together these polyhedra to obtain Z. The procedure for glueing is determined by the identifications required to reglue the original planar surfaces, giving $F_H = F_Z$, and $Z \cong M_H^3$. The correspondence is established.

Notation 5.3. For $Z \in \mathcal{PC}_3$, let $[H(Z)]$ denote the corresponding class in $\mathcal{G}_b^3$. Conversely, denote by $[Z(H)]$ the class of complexes in $\mathcal{PC}_3$ constructed from H representing a class in $\mathcal{G}_b^3$. (Thus $[Z(H)] \cong M_H^3$ unambiguously.)

Proposition 5.4. *Let $Z \in \mathcal{PC}_3$. Then*

(i) $[Z(H(Z))] = [Z]$, and $[H(Z(H))] = [H]$. *Moreover*

(ii) Z is a 3-manifold if and only if $[H(Z)] \in \mathcal{G}_r^3$.

Proof. We know that Z is a 3-manifold in the complement of its vertices. To see the structure of Z, we build it from the space $\tau_*(Z)$. The latter may be viewed as Z with a neighbourhood of each edge and vertex removed. Each edge of Z has a dual linking circle, which is a unique red circle of $F_Z = \partial\tau_*(Z)$. We obtain the closed complement of the vertices of Z by adding 2-handles to the handlebody $\tau_*(Z)$ along disjoint annuli determined by red circles on F_Z. After attaching 2-handles, we obtain a 3-manifold with some number of boundary components. Each boundary component is decomposed into a number of 2-coloured hexagons and red discs. The latter are naturally polygons: if we add a 2-handle to a red curve having t intersection points with black curves, each of the resulting discs after surgery becomes some t-gon.

If all of the components are 2-spheres, we may add balls. coning the boundary structure to the centre to obtain a closed 3-manifold. This occurs if and only if the original structure on F_Z was r-filling.

The actual structure of the link of each vertex of Z may be seen by shrinking each (red) surgery disc to a point. This gives a (possibly singular) decomposition of each 2-sphere into a union of black triangles, with vertices of degree t depending on the length of the corresponding red 'linking' circle.

Theorem 5.5. *To every decomposition of a 3-manifold as a generic polyhedron is associated a unique equivalence class of heavenly subgroups $H < \Gamma^3$. For every 3-manifold, there are infinitely many equivalence classes of heavenly subgroups giving rise to it.*

Proof. Uniqueness is just the construction above. Every 3-manifold can be triangulated in infinitely many ways.

Observation 5.6. Starting from any orientable surface F_H, we may carry out the construction just given to obtain a family of 3-dimensional singular spaces more general than those of $\mathcal{PC}_3$. These have vertices which are cones on marked surfaces. It would be interesting to find canonical 'resolutions' of these spaces, perhaps in terms of coverings of F_H.

6. Pseudo–Anosov maps

Given a marking F_M, there is canonically defined a set of invariants parametrised by triples (n, m, w), where m and n are non-zero integers of opposite sign, and w is any word in the free semi-group $< a, b : - >$ of rank 2, in which both generators appear.

Suppose F_M is oriented. It then makes sense to refer to a left or right Dehn twist about a simple closed curve. Since the curves in a class of the marking are undifferentiated, the natural assignment of integers is to give the same to each curve. To the pair (n, m) we associate the diffeomorphisms τ_b^n and τ_r^m. These twist n times (respectively m times) to the left about *all* black (respectively red) curves. A negative twist here means a twist to the right.

Penner [Pe] has shown that for n, m of *different* sign, every word as above, with $a = \tau_b^n$ and $b = \tau_r^m$, determines a pseudo–Anosov diffeomorphism of F_M. Hence there is an associated invariant, the *stretching factor* of the diffeomorphism (Thurston [Th2]).

Penner's construction uses a train track which can be constructed directly from the marking. (For train tracks arising in this way from hexagons, all complementary regions are 3-horned, and the train track is never orientable.) Knowing the sequence of intersections of the curves of one family with those of the other, labelled in any way, allows us to construct a matrix whose entries are determined by the number of times a segment of the train track wraps around any other segment under the diffeomorphism. Perron–Frobenius theory produces the stretching factor as the unique largest positive eigenvalue of this matrix.

When n and m have the same sign, Penner's construction does not apply. However, Long [Lo] has shown that there is a branched flat structure on F_M for which these maps are affine, and in fact lie in $\mathbf{SL(2;R)}$. Any word $\theta = w(\tau_b^n,\ \tau_r^m)$, with both n and m of the same sign, and which corresponds to a *hyperbolic* element of $\mathbf{SL(2;R)}$, is isotopic to a pseudo–Anosov map. The associated dilatation is then an invariant of the marking on F_M. The disadvantage of this approach is that the words which are hyperbolic for one marking may not be so for another.

Applying this to triangulations or cubings of 3-manifolds allows us to associate infinitely many computable invariants. Each is a 4-tuple, unordered, arising from a given word, two choices for which set of curves is red/black, and two for orientation of F_M. It would be of interest to know whether this numerical data has any convergence behavior under barycentric or cubical subdivision. If this is the case, we obtain correspondingly numerical invariants of the underlying manifold. The question then is to understand the change in the matrix from the train track, under subdivision.

7. 'Rubik's surfaces'

Consider the space $\mathcal{M}_g$ of equivalence classes of markings of the genus g oriented surface F, where two are equivalent if there is an orientation preserving diffeomorphism of F carrying one marking to the other, allowing the possibility of a colour change. Denote equivalence classes by $[F_M]$.

Given such a marked surface, each closed curve has a *'length'*, given by the geometric number of intersections with all curves of the other family. This induces a natural *groupoid* structure on $\mathcal{M}_g$, as follows:

Given a marked surface F_M, and curve $\sigma \in H_r \cup H_b$ of length n, cut the surface along σ, twist through an angle of $2\pi/n$ to the right, and reglue. Call this operation A_σ.

This produces a (usually) different marked surface $F_{M'} = A_\sigma(F_M)$ of the same genus. In general, the number of curves in the family not containing σ will change, as will their lengths. However, the total number of intersections is preserved, and so is the set of hexagons of F, given by cutting along all curves of both families.

In this way we construct a category, with *objects* equivalence classes $[F_M]$ of marked surfaces, and *arrows* given by the maps $\mathcal{A}_\sigma$ between them:

$$\mathcal{A}_\sigma([F_M]) = [A_\sigma(F_M)].$$

It is easy to see that this is well-defined.

These arrows are invertible, and so we obtain a *groupoid* structure on $\mathcal{M}_g$.

There are only finitely many ways in which a set of $4(g-1)$ hexagons can be identified along their boundaries to obtain F. Accordingly, this groupoid is finite.

Question 7.1. Is this groupoid connected?

The structure of each connected component of $\mathcal{M}_g$ is reminiscent of Rubik's cube. A major difference lies in the topological structure of S^2: any finite order twist about a simple closed curve extends over the complememtary regions.

The colouring of the faces of Rubik's cube has an analogue for more general families on F, since we can colour each region or even neighbourhoods of each vertex of a region a different colour. We are motivated to stay with only two families on F, since there is the clear relationship with 3-dimensional complexes. The twists correspond to redefining the identifications of faces using the rotation subgroup of the dihedral group. It is not clear how the topological type of the resulting complex changes. In general, we pass from a manifold structure to a non-manifold structure, and vice versa.

Remark 7.2. This defines a natural action on conjugacy classes of heavenly subgroups.

8. Covering spaces

Suppose H is a heavenly subgroup. Then so is any subgroup $K \subset H$.

Proposition 8.1. *If K is a finite-index subgroup of a heavenly subgroup H, then*

(i) F_K is an unbranched covering space of F_H, and

(i) M^3_K is a branched covering space of M^3_H.

Corollary 8.2. *If $K < H$ and M^3_K is a 3-manifold, then so is M^3_H.*

Proof of 8.1. Since F_H is the quotient $\mathcal{H}/H$, there is an unbranched covering projection $\mathcal{H}/K \longrightarrow \mathcal{H}/H$, which sends the marking on F_K to that of F_H. Thus we extend the covering projection over $F_K \times [-1, 1]$, and then over 2-handles.

9. Decompositions of 3-manifolds into Platonic solids

If a 3-manifold M^3 arises from a heavenly surface, we obtain a decomposition of M^3 as a union of polyhedra P_j. From the geometry of the surface, there is a natural definition of a metric on the boundary ∂P_j, by declaring each face to be a regular polygon with edges of unit length. Such a metric is singular, being Euclidean on faces and edges, but with curvature concentrated at vertices. In general, there is no realisation of such a metric as induced by an actual embedding of a polyhedron in R^3, and thus no natural extension over the 3-dimensional ball.

However, if *all* P_j happen to be the same Platonic solid, there is a natural singular metric on M^3, Euclidean across faces and in the interior of 3-balls, and with curvature concentrated along edges and at vertices.

Three of the Platonic solids lie in $\mathcal{P}_3$ – the tetrahedron, cube and dodecahedron.

Every 3-manifold occurs as a union of *tetrahedra*, *infinitely often* (take the barycentric subdivision of any triangulation, and iterate).

The two other Platonic solids serve equally nicely:

Proposition 9.1. *Every closed orientable 3-manifold can be 'cubed' (i.e. obtained by glueing cubes together).*

(After all, surely M^3 *means* $M \times M \times M$!)

Proof. Consider a tetrahedron. Add new vertices at the midpoints of each edge, the centre of each face, and at the centre of the tetrahedron. On each face, join the centre of the face to the midpoints of its boundary edges. Cone these new edges and vertices to the central vertex of the tetrahedron. This symmetrically decomposes the tetrahedron into 4 cubes.

The same construction applied to a dodecahedron decomposes the latter into 12 cubes, one for each vertex of degree 3.

Now take any closed orientable 3-manifold, with an arbitrary triangulation. Decompose each tetrahedron into four cubes, and observe that the faces are identified appropriately.

Remark 9.2. This construction canonically applies to any $X \in \mathcal{P}_3$. Thus to any $Z \in \mathcal{PC}_3$ we may associate $q(Z)$, its 'cubing'. Analogously, we may take any generic polyhedron, take a new vertex in the interior of each face, add new edges from this vertex to the vertices of the face, and cone all resulting triangles to a point in the interior of the

3-ball. This decomposes the polyhedron canonically into a union of tetrahedron. Carrying this out simultaneously for all polyhedra of a complex $Z \in \mathcal{PC}_3$ gives a new complex $\sigma(Z)$.

Lemma 9.3. *Any $Z \in \mathcal{PC}_3$ has a canonical decomposition as $q(Z)$, a union of cubes, and a canonical decomposition $\sigma(Z)$, as a union of simplices.*

Thus *two* of the Platonic solids serve as building blocks for the topological types of elements of $\mathcal{PC}_3$, in a canonical fashion.. We do not know whether this is true using dodecahedra. However, for those which are actually $manifolds$, we have the following application of the 'universal group' construction of Hilden-Lozano-Montesinos-Whitten:

Theorem 9.4 [HLMW]. *Every closed, orientable 3-manifold may be obtained by identifying faces of a finite number of dodecahedra.*

Consider now the other two Platonic solids. An octahedron has 4 edges meeting at each vertex, and we may define $\mathcal{P}_4$, $\mathcal{PC}_4$ by analogy with $\mathcal{P}_3$ and $\mathcal{PC}_3$. Define a canonical map $\theta : \mathcal{PC}_3 \longrightarrow \mathcal{PC}_4$ as follows:

Take a cube, and cone its faces to a point in its interior. This gives six square-based pyramids. If we glue two such along their square faces, we obtain an octahedron. In particular, if we glue two cubes together along a face, an octahedron is determined canonically by that face. Applying this construction to $q(Z)$, we obtain the desired element $\theta(Z)$.

The resulting complex is a union of octahedra with faces identified. Note that the link of the new vertices introduced at the centres of the cubes are again cubes. The links of the original vertices become decompositions of the 2-sphere into squares, by coning the vertices of each triangle to an interior point, and forming squares from the pair of triangles sharing a common edge.

Theorem 9.5. *Every closed, orientable 3-manifold may be obtained by identifying faces of a finite number of octahedra.*

Remark 9.6. Corresponding to $\mathcal{PC}_4$ there is a universal tesselation of the hyperbolic plane by regular, right-angled octagons. These do not give rise to such a provocative title for a paper. Similarly, corresponding to polyhedra with all vertices of degree 5, such as the icosahedron, we obtain a tesselation by regular decagons, with symmetry groups Γ^k corresponding to Γ^3. We will use $\mathcal{F}_b^k$, $\mathcal{F}_r^k$ and $\mathcal{F}^k$ according to whether polyhedra of vertex degree 3, 4 or 5 are intended, as in 3.1.

Proposition 9.7. *There is a canonical decomposition of any element $Z \in \mathcal{PC}_4$ into octahedra.*

Proof. Take $X \in \mathcal{P}_4$. Truncate all of the vertices, to obtain an element $\nu(X) \in \mathcal{P}_3$ and a number of pyramids. Now take new vertices as the midpoints of the edges of $\nu(X)$, and decompose each face as a union of squares by coning the new vertices to a point in the interior of the face. Cone all of these squares to a point in the interior of $\nu(X)$. The pyramidal truncations of the vertices now decompose into four pyramids. If now we take an element $Z \in \mathcal{PC}_4$, the pyramids fit together to produce an octahedral decomposition, giving a natural map $\mu : \mathcal{PC}_4 \longrightarrow \mathcal{PC}_4$ whose image has elements decomposed as octahedra.

Theorem 9.8. *Every closed orientable 3-manifold admits a decomposition as a union of embedded icosahedra.*

Proof. Since every closed orientable 3-manifold can be decomposed as a union of embedded octahedra, it suffices to show that an octahedron can be obtained by glueing together icosahedra.

Take two icosahedra, and on each divide the 20 triangles into two families, respectively of 15 and 5 triangles, the latter being coincident at a vertex. Identify these two icosahedra along the discs comprised of 15 triangles, leaving a ball with 10 triangles on its boundary, with 2 vertices of degree 5, and 5 of degree 4.

Take two of these objects, dividing the 10 triangles on each into a group of 6, and a group of 4, the latter being incident at a vertex of degree 4. Glue together these two polyhedra along their discs comprised of 6 triangles. The reult is an octahedron, made up of 4 *embedded* icosahedra.

Now consider a 3-manifold as a union of embedded octahedra, and view each of these as 4 embedded icosahedra. Note that the construction is non-canonical, since the interior structure of the octahedron is not symmetric.

Remark 9.9. There is an alternative construction if we content ourselves with icosahedra which may not be embedded. Again, we construct an octahedron, but this time from a single icosahedron, by identifying certain of its faces. Take a cube, and slice off all vertices by planes passing through the midpoints of edges, so as to obtain the 'truncated dual', but with red edges of length 0. The resulting polyhedron lies in $\mathcal{P}_4$, and has 8 triangles, and 6 squares.

Add diagonals to the six squares, so that parallel squares have parallel diagonals, and so that all diagonals have disjoint endpoints. The result is an icosahedron. Identify the two triangles of each square so created by folding along the corresponding diagonal. The result is an octahedron.

This procedure is *almost* canonical. There are essentially two inequivalent choices in the construction for diagonals of the squares, but after folding up to obtain an octahedron, there is a rotation carrying the result of one choice to the other. There is thus no canonical way to make a choice for each octahedron of a 3-manifold built from octahedra. This is unfortunate, since otherwise we would obtain canonical maps to $\mathcal{PC}_5$.

10. Surface structures from Platonic decompositions

A closed complex $Z \in \mathcal{PC}_k$, built from copies of the same regular Platonic solid will be called a *Platonic complex*. The surface F_Z corresponding to a Platonic complex Z has the additional property that all black curves have the same length, and conversely:

Theorem 10.1. *Let $H \in \mathcal{F}_b^k$. Then all black curves on F_H have the same length iff $Z(H)$ is Platonic. Moreover, $Z(H)$ is built respectively from tetrahedra, cubes, dodecahedra, octahedra or icosahedra iff the length n is respectively 3, 4, 5 when $k = 3$, 3 when $k = 4$, and 3 when $k = 5$.*

Proof. *Case* 1: $k = 3$, $n = 3$. After cutting along the black curves, we obtain some number of compact planar surfaces. The black boundary components are connected by a system of embedded red arcs, three connected to any given boundary component. Cutting along the red arcs produces k hexagons, with $3k$ red edges. Reglueing requires an even number of identifications, and so $k = 2m$. This gives a surface with $2m$ black boundary components and Euler characteristic $-m$. Since adding a disc to each boundary component produces a sphere, $-m + 2m = 2$, and so there are four hexagons glued together. The structure is that of a tetrahedron, with vertices truncated.

Case 2: $k = 3,\ n = 4$. Proceeding as above, our planar surface has $4n$ hexagons, since the number of black edges is divisible by 4. The Euler characteristic gives $3n + 4n - 6n = 2$, and so there are eight hexagons and six black squares. We have a truncated octahedron, and the octahedron is dual to a cube.

Case 3: $k = 3,\ n = 5$. We obtain $10n$ hexagons, such that $6n + 10n - 15n = 2$. This gives the dodecahedron.

Case 4: $k = 4,\ n = 3$. The planar surfaces have $3k$ octagons, giving $4k$ boundary components. Thus $4k + 3k - 6k = 2$, giving 6 octagons fitting together as a truncated cube dual to an octahedron.

Case 5: $k = 5,\ n = 3$. The planar surfaces have 12 decagons fitting together as a truncated dodecahedron dual to an icosahedron.

Remark 10.2. There are considerable restrictions on the combinatorics of subgroups of Γ^k, on the one hand as to whether the resulting red or black circles can fill, and on the other, what possible lengths may arise. For filling families of black curves, the lengths quoted are the only ones possible, if all lengths are equal.

11. Singular geometry of 3-manifolds

Thurston and Weeks have given a delightful introduction to the topology and geometry of two and three dimensional manifolds in [T-W]. We highly recommend the reading of their article in conjunction with this section, particularly because of their Figures.

The Platonic solids have a canonical geometry. Glueing copies together to obtain a closed 3-manifold gives rise to a *singular geometry* on the 3-manifold, which in the case of cubes gives rise to unexpected but significant results ([A-R]). We describe three simple examples to illustrate how these singular metrics capture the notions of zero, negative and positive curvature.

Example 11.1 – Flat geometry. The 3-torus.

Some of Thurston and Weeks' examples arise as the configuration spaces of certain mechanical linkages. A triple crank has configuration space the 3-dimensional torus $T^3 \cong S^1 \times S^1 \times S^1$, whereas three double cranks with a central pin gives rise to a genus three surface. This configuration space has a natural decomposition into eight hexagons glued together, and by declaring each hexagon to be right-angled, regular and hyperbolic, Thurston and Weeks demonstrate the naturality of negative curvature in two dimensions. This example has the following alternative description:

The 3-torus $T^3 \cong S^1 \times S^1 \times S^1$ can be obtained by identifying opposite faces of a unit cube by parallel translation. The corresponding heavenly surface arises from the 8 hexagons of the truncated octahedron, giving a genus 3 Heegard decomposition of T^3, marked with red and black circles. The picture is exactly that of [T-W], page 102, with the additional 2-colouring.

Observe that all red and black circles have 4 intersection points. We refer to this as the 'length' of each curve. For the black circles, this corresponds to the faces of the cube being squares. For the red, it means edges of the cube are identified in sets of four. Since the natural Euclidean geometry of the cube gives a dihedral angle of $\pi/2$ at each edge, the flat geometry extends over the edges, and also over the vertices of T^3. The singular geometry is in fact non-singular, and corresponds to the canonical flat metric on T^3.

The next two examples are related by 'Rubik's twists':

Example 11.2 – Negative curvature. The Weber-Seifert space [WS].

Take a single dodecahedron, and identify opposite faces after a rotation of $3\pi/5$, always in the same sense. This gives a manifold, which admits a complete metric of constant curvature -1.

The corresponding heavenly surface is of genus 6, all black edges have length 5, and similarly all red edges have length 5.

A singular metric arises by taking the Euclidean metric on the dodecahedron. Since edges are identified 5 at a time, there will be singularities in the metric, with negative curvature concentrated along edges, and at the vertex.

An alternative singular metric arises by 'cubing' the dodecahedron first, as was done above for general polyhedra in $\mathcal{P}_3$, and then declaring all cubes to be regular. This leads to a decomposition of the Weber-Seifert space into 20 cubes. The corresponding heavenly surface has all black circles length 4, all red circles of length 4 or 5. The singular geometry extends as flat geometry over those edges corresponding to red circles of length 4, but concentrates negative curvature along the remaining edges and vertices.

We leave as an exercise for the reader the drawing of this heavenly surface, of genus 60. (This is not too hard.)

Example 11.3 – Positive curvature. Poincaré's homology sphere.

Again, take a single dodecahedron, and identify opposite faces after a rotation of $\pi/5$, always in the same sense. This gives a manifold, which admits a complete metric of constant curvature $+1$ (c.f. Weber-Seifert!).

The corresponding heavenly surface is of genus 6, all black edges have length 5, but all red edges have length 3.

A singular metric arises by taking the Euclidean metric on the dodecahedron. Since edges are identified 3 at a time, there will be singularities in the metric, with positive curvature concentrated along edges, and at the vertex, due to an angle deficit.

The heavenly surface is obtained from that of the last example by twisting along the black circles.

We also obtain an alternative singular metric by 'cubing' the dodecahedron, into 20 cubes. The corresponding heavenly surface has all black circles length 4, all red circles of length 3, 4 or 5. The singular geometry extends as flat geometry over those edges corresponding to red circles of length 4, concentrates negative curvature along those of length 5, and positive along those of length 3.

It is not clear whether the corresponding heavenly surface is obtained from that of the last example by twisting along circles.

12. Minimal surfaces and non-compact 3-manifolds

Theorem 12.1. *Every non-compact 3-manifold arises from a subgroup of* Γ^3 *of infinite index.*

Proof. Every 3-manifold can be triangulated. Applying the above construction yields a non-compact surface of infinite genus which is heavenly.

The tesselation of *non-compact* Euclidean 3-space by unit cubes can be studied from this view point: The truncation of the dual of a cube is a truncated octahedron. Glueing these together, we obtain one $half$ of the familiar tesselation of $\mathcal{R}^3$ by truncated octahedra. The boundary of the resulting infinite handlebody has hexagonal structure, which we have observed has a natural hyperbolic structure. (A picture of this surface appears in Coxeter [Co1].)

If instead of a *hyperbolic* regular geometry on each hexagon, we use the standard Euclidean structure, all curvature becomes concentrated at the vertices. These are then points of negative curvature, and the structure should be interpreted as a *P.L. minimal surface*. There is a symmetry between the black and red families – all have length 4. Observe that adding 2-handles along the red curves creates infinitely many spheres: all of these can be filled in to give $\mathbf{R^3}$. It seems a very difficult problem to identify the structure of the *ends* of any non-compact 3-manifold constructed in this fashion, such as with a triangulation or cubing of the Whitehead manifold.

This surface in $\mathbf{R^3}$ is topologically close to the standard triply-periodic smooth minimal surface. The smooth surface is naturally decomposed by geodesics into hexagons, giving the same combinatorial structure as that derived above.

In many situations – such as for the standard metric on S^3 – there is a Heegard surface for each genus which is actually a minimal surface. This suggests that the marked structure arising from a polyhedral decomposition of an arbitrary 3-manifold carries the structure of a P.L. minimal surface with respect to some singular Euclidean metric.

The question presenting is then: is it possible to perturb some canonical singular metric to a smooth one, while simultaneously deforming the surface and its markings, such that the latter remains minimal, as a *smooth* surface, with the markings geodesic? This seems highly plausible.

This and other questions will be addressed elsewhere ([A-R]).

13. Final remark

We have demonstrated a representation of all possible 3-dimensional spaces in terms of Escher's *Heaven and Hell*. There is an amusing parallel of such a representation with the world view of the scientist-turned-mystic, Emanuel Swedenborg (1688-1772), whose purported abilities ranged from being one of the originators of the nebular theory of the origin of the planets, the engineer of the largest dry-dock of his time, through to announcing the day of his own death months in advance, by way of apology in a letter to John Wesley for being unavailable to meet.

Swedenborg's vision of reality was that of an interlocking symmetry: the Human world occupying a neutral level, the free meeting place of three levels each of Heaven and Hell. This is described in one of his thirty-two religious volumes, titled **Heaven and Hell**. Kant, the only known purchaser of one of the four copies of Swedenborg's writings on mysticism known to have been sold, considered the system 'fantastic', although 'perhaps no more so than orthodox metaphysics' ([Ru] p.679). Nonetheless, Swedenborg's views retain a following, and there have even been interesting, albeit metaphysical, descriptions of the experience of many schizophrenics which are not in discord with his scenario ([VD]).

Although Escher, in his writings, makes no mention of Swedenborg, it is conceivable that this Fuchsian group of symmetries was chosen in order to depict three levels each of Heaven and Hell, represented by the six figures in each hexagon. It is likely that Escher was aware of the work of William Blake, who in turn was a 'solitary Swedenborgian' ([Ru], p.655). It is clear that Escher was not averse to using religious symbolism, and did not limit his inspiration to standard Christian imagery.

In the past, arguments have been generated by serious proposals concerning the extent to which Escher studied the psychology of perception. It is not our intention to create yet another controversy concerning things influential in Escher's thinking. Although Escher was inspired by the geometry of Coxeter's papers, he never felt comfortable with the underlying mathematics ([Es3], 91-93). Thus it is a remarkable coincidence that, of all his works utilising concepts of mathematics, the one he chose to call Heaven and Hell should be this one! No doubt this would have amused Swedenborg, whose first published work was mathematical– a treatise on algebra, the first in Swedish on the subject. It certainly amuses us – but we do not attach any great significance to the comments made above, since the first exposure Escher had to non-Euclidean tesselations was through diagrams of Coxeter, which included those with 2-fold and 3-fold points of symmetry.

14. Conclusion

It might be tempting to feel that our simple observation on the relationship between heavenly surfaces and 3-dimensional spaces might lead to some simplifications in our understanding of the latter. Although some amusing aspects have emerged, the viewpoint better adopted is that the symmetries of Heaven and Hell have far greater complexity than may appear at first sight.

Singular geometry of 3-manifolds, at this stage at least, offers a much richer line of research.

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